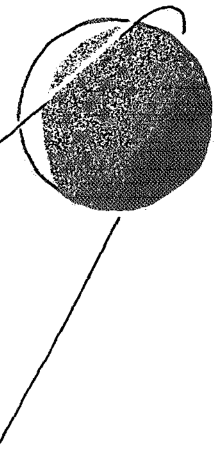
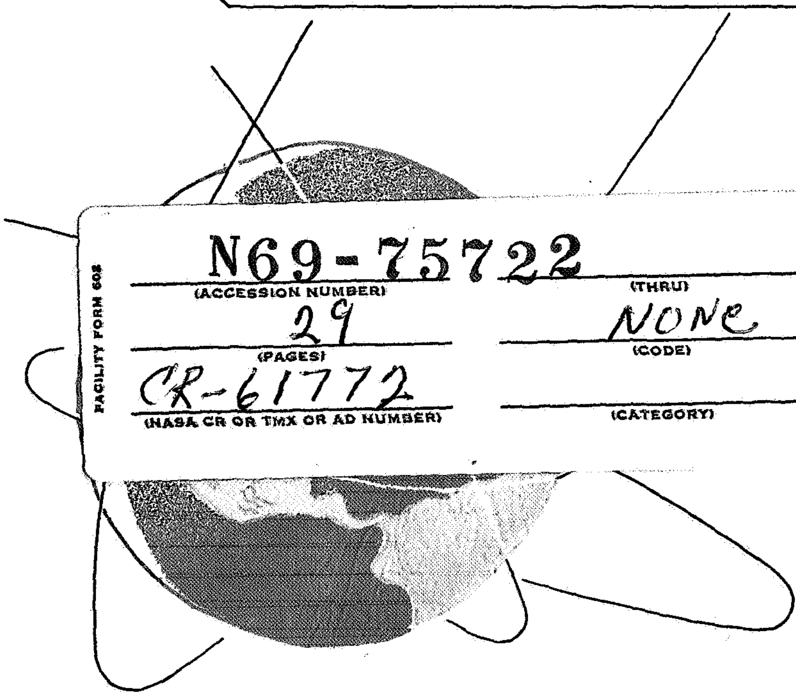


SOME APPROACHES TO SUBOPTIMAL FEEDBACK CONTROL OF NONLINEAR SYSTEMS

BY

W. L. GARRARD, N. H. McCLAMROCH, L. G. CLARK

RM EMRL 1028 MAY 1967



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W.L. Garrard*, N.H. McClamroch*
and L.G. Clark*

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ABSTRACT

A method is developed for the determination of suboptimal control laws for nonlinear dynamical systems. The control laws determined by use of this method are in time invariant, feedback form and approximately minimize a performance index which is the integral of a positive definite function of the state plus a quadratic function of the control. The basis of the proposed technique is a method for the determination of approximate solutions for the associated Hamilton-Jacobi-Bellman equation. The method is applied to three examples, and the results are shown to compare favorably with those obtained by use of other suboptimal control procedures. The method developed in this paper is applicable, in a practical sense, to systems of higher than second order and seems to hold promise as a means for solving a large class of optimization problems.

SOME APPROACHES TO SUBOPTIMAL FEEDBACK CONTROL OF NONLINEAR SYSTEMS

1. INTRODUCTION

Many studies concerning systems which are optimal in some sense have appeared in the last several years. However, much of the literature has dealt with theoretical aspects of the optimal control problem, and comparatively few direct applications of the theory to the synthesis of feedback control systems have been presented. The purpose of this paper is to present the results of a study of a class of optimization problems directly related to the design of regulator type control systems.

The synthesis of approximately optimal feedback controls for dynamical systems governed by ordinary nonlinear differential equations is considered. For most nonlinear systems it is impossible to analytically determine the exact optimal control law; consequently, the designer is faced with the task of synthesizing a suboptimal control. Several methods for determining approximately optimal feedback control laws for nonlinear systems have been proposed. The advantages and limitations of several of the more successful of these methods are discussed and a new technique which in certain cases appears to possess advantages over existing methods is presented.

2. PROBLEM STATEMENT

This study is limited to controllable dynamical systems governed by

$$\dot{x} \equiv \frac{dx}{dt} = f(x) + Bu, \ f(0) = 0 \qquad (1)$$

$$x(0) = x_0$$

where the x, the state, is an n-vector and u, the control, is an m-vector. The constant vector, x_0, is the initial state of the system, and B is a $m \times n$ matrix of constants. The optimal control problem associated with this dynamical system is the determination of the control which will transfer the initial state to some specified terminal set about the origin, denoted by S, and also minimize the integral performance criterion

$$J(x_0,u,S) = \int_0^T [g(x) + u'Ru]dt \qquad (2)$$

where $g(x)\ 0$, $x \neq 0$ and $g(0) = 0$, and the $m \times n$ matrix R is symmetric and positive definite. A prime denotes the transpose of a matrix.

The suboptimal control problem associated with the dynamical system defined by (1) and the performance criterion (2) is the determination of a control which satisfies the following properties: the control must transfer the initial state to S, the control is a function of the present state of the system only; i.e., it is in a time-invariant, feedback form, and the control is close to the optimal in some sense for a set of initial states. In the present work a control is considered close to the optimal if it approximately satisfies the conditions which the optimal control must satisfy. Obviously there may be many controls which satisfy the above criteria; however, from a practical point of view, this is an advantage rather than a limitation since a control can be chosen which best satisfies criteria such as ease of implementation, reliability, and other important design criteria.

3. OPTIMAL CONTROL

Three techniques which have received the most attention are the calculus of variations, Pontryagin's maximum principle, and the Hamilton-Jacobi-Bellman approach. Under appropriate assumptions on (1) and (2), each of these three methods gives enough information to determine the optimal control. In what follows we shall consider only the latter two approaches since the calculus of variations and the maximum principle are similar in application.

The minimum (or maximum) principle, as developed by Pontryagin, et. al. [1], gives necessary conditions for the optimal control. The optimal control, u^*, is characterized by the existence of a vector, p, such that the Hamiltonian function

$$H(x,p,u) = g(x) + u'Ru + p'f(x) + p'Bu \qquad (3)$$

is a minimum when $u(t) = u^*(t)$. In addition, the state x and the vector p satisfy the following set of differential equations

$$\dot{x} = \frac{\partial H}{\partial p}(x,p,u^*) \ , \quad x(0) = x_0 \qquad (4)$$

$$\dot{p} = -\frac{\partial H}{\partial x}(x,p,u^*) \ , \quad p(T) \text{ is normal to } S \ . \qquad (5)$$

Since the boundary conditions on (4) and (5) are specified at different times, solution of these equations may be difficult, even by numerical means.

A generalization of the classical Hamilton-Jacobi theory provides a somewhat different approach to the optimization problem, and gives sufficient conditions for optimality. If the minimum of the Hamilton function (3) with respect to u occurs when $u = k(x,p)$ and is denoted by $H(x,p)$, and if there is a twice continuously differentiable function, $V(x,t)$, satisfying

$$\frac{\partial V}{\partial t} + H(x,\frac{\partial V}{\partial x}) = 0 \ , \quad V = 0 \text{ on } S \qquad (6)$$

($\frac{\partial V}{\partial x}$ denotes the gradient of $V(x,t)$), then the optimal control is given by

$$u = k(x,\frac{\partial V}{\partial x}) \ . \qquad (7)$$

Thus, it is necessary to solve a first order, partial differential equation which is in general nonlinear. Discussions of this approach are found in [2,3,12].

4. SUBOPTIMAL CONTROL

The optimal control obtained by either of the above approaches will rarely be in time-invariant feedback form, and in most cases analytical computation of the optimal control law is impossible. Thus, the designer is faced with the determination of a suboptimal control. The methods which have been proposed in the literature for obtaining solutions to the suboptimal control problem outlined above are essentially methods for obtaining approximate solutions to the equations corresponding to either (4) and (5), or (6). Several representative methods which have been used to obtain suboptimal control laws satisfying the desired properties are discussed. The methods which are considered are (a) linearization, (b) parameter optimization, (c) equivalent linearization, and (d) perturbation.

<u>a. Linearization</u> The non-linear dynamical system (1) is approximated by a linear system of the form

$$\dot{x} = Ax + Bu \qquad x(0) = x_0 \tag{8}$$

where A is a $n \times n$ constant matrix. The performance criteria (2) is approximated by

$$J(x_0,u,s) = \int_0^T (x'Qx+u'Ru)dt \tag{9}$$

where Q is a constant $n \times n$, symmetric, positive-definite matrix. The optimal control for (8) and (9) can be determined in feedback form by using either the maximum principle or the Hamilton-Jacobi-Bellman equation. The resulting control is given in [4] as

$$u = -R^{-1}B'Px \tag{10}$$

where P satisfies

$$\dot{P} + PA + A'P - PBR^{-1}B'P + Q = 0 \ , \quad P(T) = 0 \ . \tag{11}$$

However, P is a function of time, and thus u is not time invariant. This control can be made time invariant in two ways. First, as T approaches infinity, it can be shown that $\dot{P}$ approaches zero; therefore, if T is large the solution to the algebraic matrix equation

$$PA + A'P - PBR^{-1}B'P + Q = 0 \tag{12}$$

can be used as the constant matrix P in (10). Second, the solution of (11), $P(t)$ can be averaged over the control interval $[0,T]$ in the following manner

$$\bar{P} = \frac{1}{T} \int_0^T P(t)dt \ . \tag{13}$$

Thus, the control obtained by using a constant matrix, determined by either (12) or (13), may be employed as a suboptimal control for the nonlinear system (1) and (2).

b. Parameter Optimization. Several parameter optimization schemes have been proposed. One approach [5] has been to assume a fixed form for the control containing an arbitrary constant vector b, i.e.,

$$u = k(x,b) \tag{14}$$

and then determine the value of b for which (2) is minimized. Using the minimum principle, one must solve the equations corresponding to (4) and (5) for some fixed x_o.

A similar approach has been considered in [6]. However, instead of directly choosing a form for the control of a solution to equation (6), denoted by $V(x,b)$, is assumed which involves an unknown constant vector b. Several methods are suggested in [6] for choosing b so that (6) is nearly satisfied. Again the choice of b depends on a particular initial state. Both of these parameter optimization methods have the disadvantages that the problem of determining the minimizing value of b is nearly as difficult as the original problem, and the value of b depends on the initial state.

c. Equivalent Linearization. This approach is based on the fact that the linear problem with quadratic performance criteria (8) and (9), can be solved exactly as indicated in (a).

The nonlinear system (1) is approximated by

$$\dot{x} = A(x)x + Bu \ , \qquad x(0) = x_0 \ , \tag{15}$$

performance criteria (2) is approximated by

$$J(x_0,u,S) = \int_0^\infty [x'Q(x)x + u'Ru]dt \tag{16}$$

where Q is symmetric. A suboptimal control is obtained from (10) A, Q, P are not constant but are functions of x. Thus,

$$u = -R^{-1}B'P(x)x \tag{17}$$

where $P(x)$ satisfies

$$P(x)A(x) + A'(x)P(x) - P(x)BR^{-1}B'P(x) + Q(x) = 0 \ . \tag{18}$$

An application of this approach is given in [7].

<u>d. Perturbation</u>. Another approach for obtaining a suboptimal control is by use of an approximate solution of the Hamilton-Jacobi-Bellman equation by means of power series expansion [8]. The functions $f(x)$ and $g(x)$ are expanded in a power series in x about some point, usually $x = 0$, giving

$$\dot{x} = Ax + \sum_{n=2}^{\infty} \varepsilon^{(n-1)} f_n(x) + Bu \tag{19}$$

$$J[x_0,u,S] = \int_0^{\infty} [x'Qx + \sum_{n=3}^{\infty} \varepsilon^{(n-2)} g_n(x) + u'Ru]dt \tag{20}$$

where $f_n(x)$ is a vector with components which are polynomials in x of degree n, and $g_n(x)$ is also a polynomial in x of degree n. The parameter ε has been introduced for bookeeping convenience.

After minimization of the Hamiltonian, the Hamilton-Jacobi-Bellman equation is

$$\frac{\partial V'}{\partial x}[Ax + \sum_{n=2}^{\infty} \varepsilon^{n-1} f_n(x)] - \frac{1}{4}(\frac{\partial V}{\partial x})' BR^{-1}B' \frac{\partial V}{\partial x} + x'Qx$$

$$+ \sum_{n=3}^{\infty} \varepsilon^{n-2} g_n(x) = 0 \quad . \tag{21}$$

A formal power series expansion for $V(x)$ is assumed, that is

$$V(x) = \sum_{n=2}^{\infty} \varepsilon^{(n-2)} V_n(x) \tag{22}$$

where $V_n(x)$ denotes a general polynomial in x of degree n with coefficients as yet unknown. A recurrence relation for the unknown coefficients in (22) is generated by substituting (22) into (21) and equating the coefficients of ε^n to zero. The resulting suboptimal control for some set of initial states is given by

$$u = -\frac{1}{2}R^{-1}B' \left[\sum_{n=2}^{\infty} \varepsilon^{(n-2)} \frac{\partial V_n(x)}{\partial x} \right] . \tag{23}$$

5. A NEW METHOD

This method, called the combination method, is based on methods (c) and (d), and seems to have certain practical advantages over the methods previously discussed. The basic procedure consists of approximating (1) and (2) by

$$\dot{x} = Ax + \varepsilon f(x) + Bu , \tag{24}$$

and

$$J(x_0,u,S) = \int_0^T [x'Qx + \varepsilon g(x) + u'Ru]dt . \tag{25}$$

Again, the basic idea is to obtain a "good" approximate solution to the Hamilton-Jacobi-Bellman equation by expanding $V(x)$ in a power series as in (22). The equation to be solved is

$$\frac{\partial V}{\partial t} + \frac{\partial V'}{\partial x}[Ax + \varepsilon f(x)] - \frac{1}{4}\frac{\partial V'}{\partial x}BR^{-1}B'\frac{\partial V}{\partial x} + x'Qx + \varepsilon g(x) = 0 . \tag{26}$$

Assuming a formal power series expansion as in (22) and equating powers of ε to zero,

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2'}{\partial x} Ax - \frac{1}{4} \frac{\partial V_2'}{\partial x} BR^{-1}B' \frac{\partial V_2}{\partial x} + x'Qx = 0$$

$$\frac{\partial V_3}{\partial t} + \frac{\partial V_3'}{\partial x} Ax + \frac{\partial V_2'}{\partial x} f(x) - \frac{1}{4} \frac{\partial V_2'}{\partial x} BR^{-1}B' \frac{\partial V_3}{\partial x}$$

$$- \frac{1}{4} \frac{\partial V_3'}{\partial x} BR^{-1}B' \frac{\partial V_2}{\partial x} + g(x) = 0 \qquad (27)$$

$$\vdots$$

$$\frac{\partial V_n}{\partial t} + \frac{\partial V_n'}{\partial x} Ax + \frac{\partial V_{n-1}'}{\partial x} f(x) - \frac{1}{4} \sum_{\substack{k \geq 2 \\ \ell \geq 2}}^{n+2} \frac{\partial V_k'}{\partial x} BR^{-1}B' \frac{\partial V_\ell}{\partial x} = 0 \ .$$

where in the summation $k + \ell = n + 2$.

Writing $f(x) = C(x)x$ and $g(x) = x'D(x)x$, where $D(x)$ is symmetric, the previous equations become

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2'}{\partial x} Ax - \frac{1}{4} \frac{\partial V_2'}{\partial x} BR^{-1}B' \frac{\partial V_2}{\partial x} + x'Qx = 0$$

$$\frac{\partial V_3}{\partial t} + \frac{\partial V_3'}{\partial x} Ax + \frac{\partial V_2'}{\partial x} C(x)x - \frac{1}{4} \frac{\partial V_2'}{\partial x} BR^{-1}B' \frac{\partial V_3}{\partial x}$$

$$- \frac{1}{4} \frac{\partial V_3'}{\partial x} BR^{-1}B' \frac{\partial V_2}{\partial x} + x'D(x)x = 0 \ , \qquad (28)$$

$$\vdots$$

$$\frac{\partial V_n}{\partial t} + \frac{\partial V_n'}{\partial x} Ax + \frac{\partial V_{n-1}'}{\partial x} C(x)x - \frac{1}{4} \sum \frac{\partial V_k'}{\partial x} BR^{-1}B' \frac{\partial V_\ell}{\partial x} = 0 \ .$$

In order to determine $V_2, V_3, \ldots, V_n, \ldots$ we solve these equations successively, treating the matrices $C(x)$ and $D(x)$ as constants. Each equation in (28), with the exception of the first, is a first-order, linear, partial differential equation. Although the first equation is not linear, it is the same equation as one would obtain by solving (8) and (9) using the Hamilton-Jacobi-Bellman equation. In order to solve (28), assume

$$V_n = x'M_n x \tag{29}$$

where M_n is a $n \times n$ symmetric matrix, to be determined. This leads to the matrix equations

$$\begin{aligned}
&\dot{M}_2 + M_2A + A'M_2 - M_2BR^{-1}B'M_2 + Q = 0 \\
&\dot{M}_3 + M_3(A-BR^{-1}B'M_2) + (A'-M_2BR^{-1}B')M_3 \\
&\qquad\qquad = -M_2C-C'M_2-D \qquad (30) \\
&\qquad\vdots \\
&\dot{M}_n + M_n(A-BR^{-1}B'M_2) + (A'-M_2BR^{-1}B')M_n = Q_n
\end{aligned}$$

with $M_i(T) = 0$ for $i = 2, 3, \ldots, n, \ldots$

and $Q_n = \sum_{\substack{k \geq 3 \\ \ell \geq 3}}^{n+2} M_k BR^{-1}B'M_\ell - M_{n-1}C-C'M_{n-1}$, $k+\ell = n+2$.

The problem of obtaining matrices $M_2, M_3, \ldots, M_n, \ldots$ which are not explicit functions of time and which "nearly" satisfy (30) can be approached in two ways. First, Kalman [4] has shown that the first equation of (30) has a unique positive definite solution $M_2(t)$. For $n \geq 3$, (30) is a set of linear differential equations and thus, for each n, $M_n(t)$ exists and is unique. In order to determine constant matrices which are close approximations to $M_n(t)$ for each n, the following technique can be used:

$$\bar{M}_k = \frac{1}{T} \int_0^T M_k(t)dt \quad , \quad k = 2, 3, \ldots \quad . \tag{31}$$

Second, if the final time, T, is sufficiently large, we can assume that $\dot{M}_n = 0$ for each n and (30) represents a set of algebraic equations. Again, Kalman has shown that for $\dot{M}_2 = 0$, the first equation of (30) has a unique solution. In addition, the remaining equations in (30) have a unique solution for each n as demonstrated by Lefschetz [9]. Lefschetz also shows that if Q_n is positive [negative] definite, then M_n is negative [positive] definite. Thus, by computing Q_n it is possible to determine the effect of M_n without calculating this matrix.

Using either of the above approaches, a suboptimal control is

$$u = -\frac{1}{2}R^{-1}B' \sum_{n=2}^{\infty} \varepsilon^{(n-2)} \frac{\partial}{\partial x} [x'M_n(x)x] \quad . \tag{32}$$

In general M_n is a function of the state; therefore, u will usually be a nonlinear function of the state.

6. ILLUSTRATIVE EXAMPLES

Since the details, advantages, and limitations of the various schemes are best illustrated by examples, three nonlinear systems are analyzed. Suboptimal control laws are derived by use of the various methods outlined above, and the results obtained by use of these laws are compared. The parameter optimization schemes were tested using initial conditions different from those for which the control law was originally computed. This was done in order to see how well the control laws calculated by parameter optimization perform for a variety of initial conditions.

<u>Example 1</u>: Consider the system described by

$$\dot{x} = u - x^2 \tag{33}$$

where u is to be an acceptable solution to the suboptimal control problem associated with the minimization of $J = \frac{1}{2}\int_0^{1.0}(x^2+u^2)dt$ for the target set $S = R$, the real line:

<u>Linearization</u> Considering only the linear part of equation (33), $\dot{x} = u$, and the performance index $J = \frac{1}{2}\int_0^{1.0}(x^2+u^2)dt$. From (10), $u = -2P(t)x$ where from (11)

$$P(t) = -\frac{1}{2}\left[\frac{1-e^{8(1-t)}}{1+e^{8(1+t)}}\right] ;$$

averaging $P(t)$ as in (13), $\bar{P} = 0.345$, and therefore

$$u = -0.690x \; . \tag{34}$$

Parameter Optimization. The above problem is considered in [5]. The form of the control is assumed to be linear, $u = ax$, where a is an unknown constant. For an initial condition $x(0) = 10.0$, the control is found to be $u = -0.1038x$.

Equivalent Linearization. The differential equation describing the system is rewritten

$$\dot{x} = u - \alpha x, \; \alpha = x \; . \tag{35}$$

This equation is of the same form as (15) where $A = -\alpha$ and $B = 1$. If $R = Q = \frac{1}{2}$, J is the same as (16); consequently, from (17) and (18)

$$u = -[\sqrt{x^2+1}-x]x \; . \tag{36}$$

Perturbation. Equation (33) may be recast in the form of (19) by noting that $A = 0$, $B = 1$, $f_2(x) = -x^2$, $f_n = 0$ for $2<n$, $g_n = 0$, $Q = \frac{1}{2}$, and $R = \frac{1}{2}$. Substituting the $V(x)$ from

(22) into (21) and equating the coefficients of powers of ε to zero, the following equations are obtained

$$-\frac{1}{2}\left(\frac{\partial V_2}{\partial x}\right)^2 + \frac{x^2}{2} = 0$$

$$\frac{\partial V_2}{\partial x} x^2 - \frac{\partial V_2}{\partial x} x^2 - \frac{\partial V_2}{\partial x}\frac{\partial V_3}{\partial x} = 0 \tag{37}$$

$$\vdots$$

Letting $V_2(x) = a_2 x^2$, $V_3 = a_3 x^3$, from equations (37) $a_2 = \frac{1}{2}$, $a_3 = -\frac{1}{3}$, thus, from (23) the control is

$$u = -x + x^2 \ . \tag{38}$$

<u>Combination Method</u>. The above solution may be improved by rewritting (33) as

$$\dot{x} = u - \varepsilon \alpha x \ , \quad a = x \ . \tag{39}$$

The first two equations of (28) are

$$\frac{\partial V_2}{\partial t} - \frac{1}{2}\left(\frac{\partial V_2}{\partial x}\right)^2 + \frac{x^2}{2} = 0 \tag{40}$$

$$\frac{\partial V_3}{\partial t} - \frac{\partial V_2}{\partial x} dx - \left(\frac{\partial V_2}{\partial x}\right)\left(\frac{\partial V_3}{\partial x}\right) = 0 \quad . \tag{41}$$

Assuming $V_2 = M_2x^2$ and $V_3 = M_3x^2$ and substituting into (40) and (41),

$$\dot{M}_2 - 2M_2^2 + \frac{1}{2} = 0 \ , \quad M_2(1) = 0 \ , \tag{42}$$

$$\dot{M}_3 - 2M_2\alpha - 4M_2M_3 = 0 \ , \quad M_3(1) = 0 \ . \tag{43}$$

Solving (42) for $M_2(t)$ and averaging as in (31), $\bar{M}_2 = 0.345$. Substituting $\bar{M}_2$ into (43) and solving

$$M_3(t) = 0.5\alpha(e^{1.38(t-1)}-1)$$

and averaging $M_3(t)$ over $[0,1]$, $\bar{M}_3 = -\ 0.105\alpha$. From (32), the suboptimal control for $\varepsilon = 1$ is

$$u = -0.69x + 0.315x^2 \ . \tag{44}$$

In order to determine the effectiveness of the above suboptimal control laws, a digital computer simulation of the system was carried out. The results are summarized in Table 1. It should be noted that the performance of the various methods depends largely upon the initial condition on x. For an initial condition of 0.5, the control determined by the perturbation method gives a substantially smaller value to the performance index then the control determined by the parameter optimization method; however, for an initial condition of 5.0, the parameter optimization method gives much better results than the perturbation method. The equivalent linearization, combination, and linearization methods all give good results. Choice of a control law for this problem would probably be made from one of these three methods depending upon the range of initial conditions expected and upon the form of control desired.

Typical solutions are compared in Fig. 1. In Fig. 1a trajectories are plotted versus time and it is seen that all the trajectories are close to one another; however, in Fig. 1b where the various control laws are plotted versus time, the control laws are shown to vary considerably.

TABLE 1

Method	Control Law	Performance Index Initial Conditions 0.5	5.0
Parameter Optimization	$-0.1038x$	.0870	3.2220
Equivalent Linearization	$-[\sqrt{x^2+1}-x]x$	.0772	2.0461
Perturbation	$-x + x^2$	.0762	47.9102
Combination	$-0.69x + 0.315x^2$	.0741	2.6787
Linear	$-0.69x$	.9760	2.5104

Fig. 1a

STATE vs TIME

$\dot{X} = U - X^2$

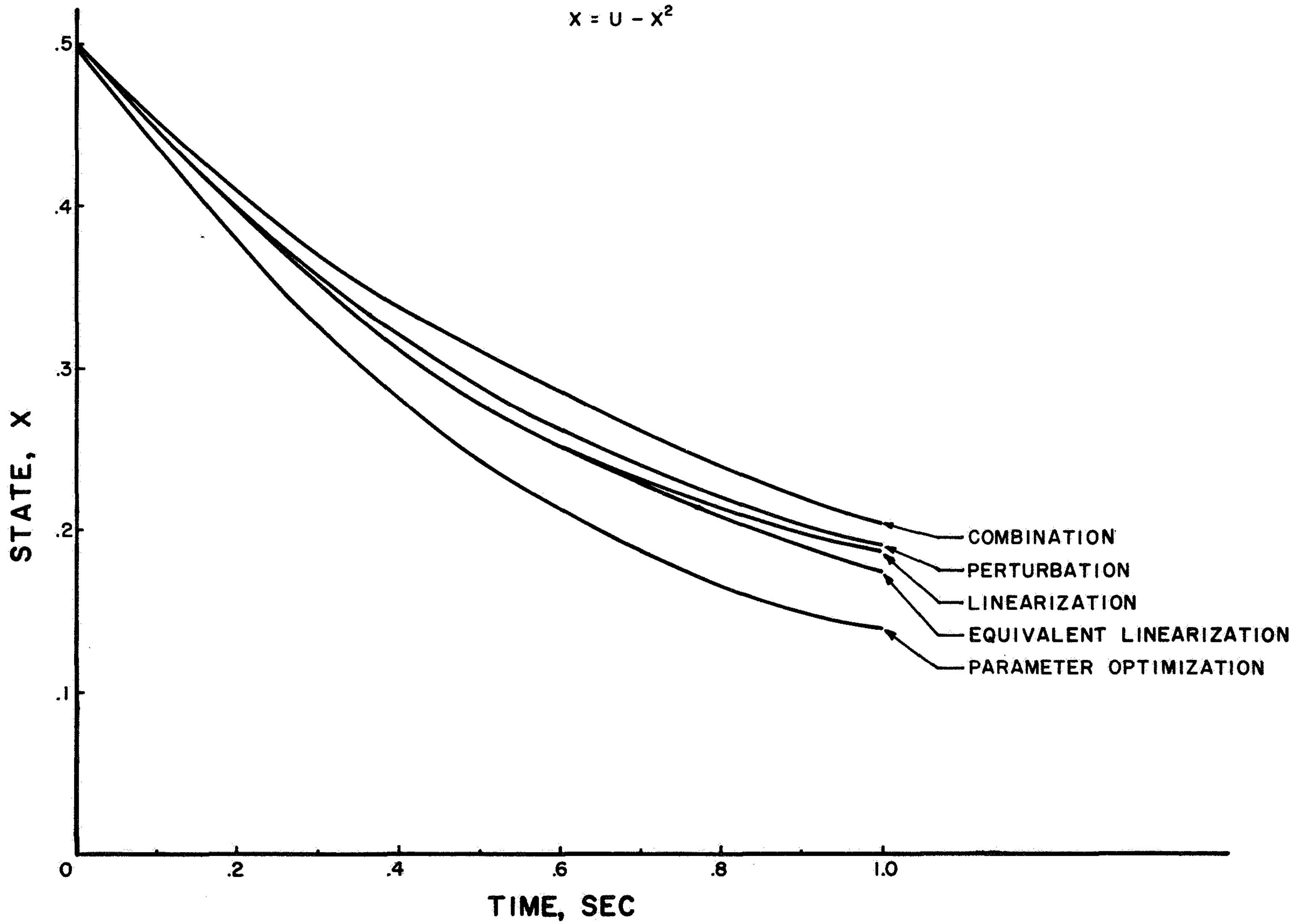
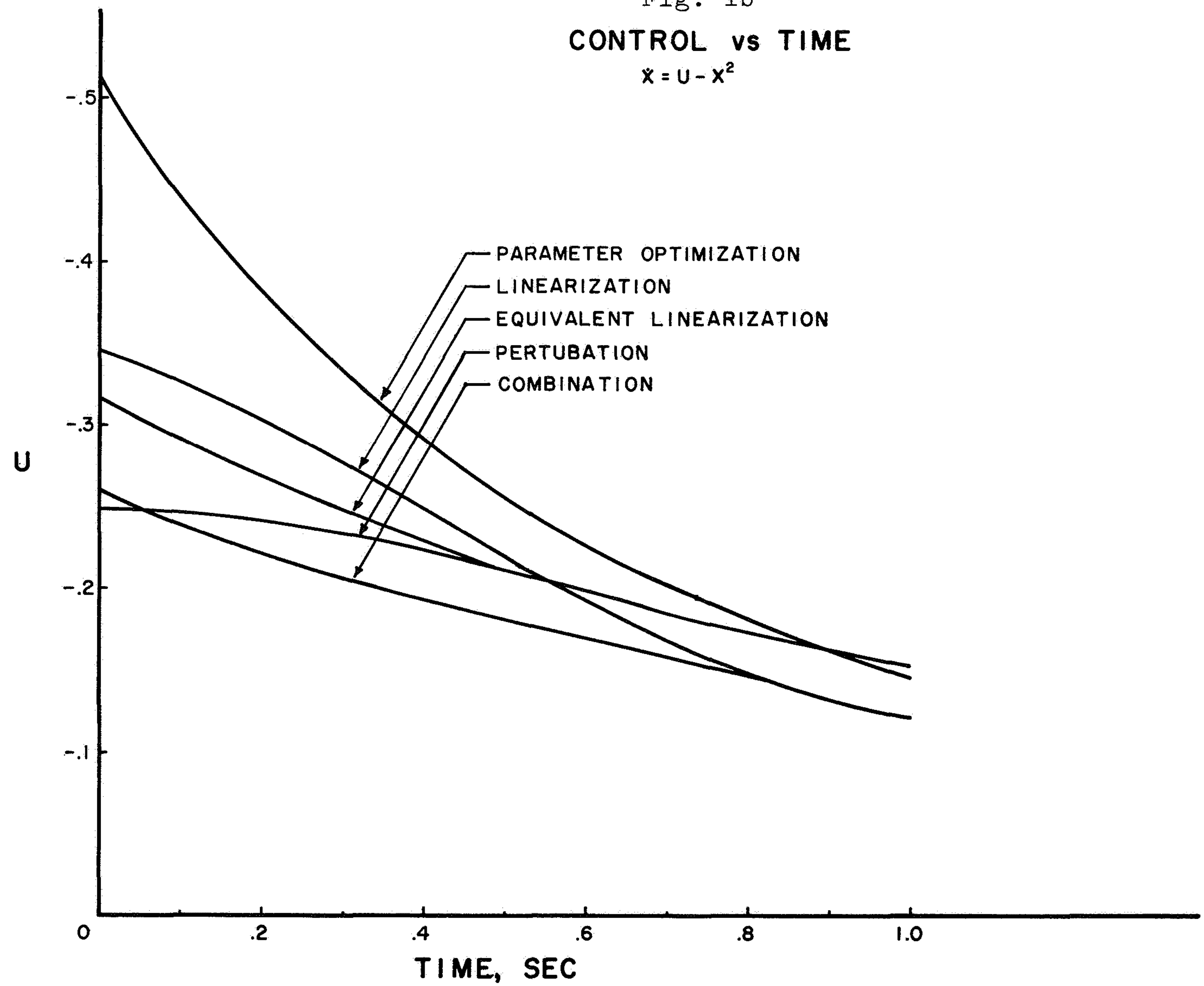


Fig. 1b

CONTROL vs TIME

$\dot{X} = U - X^2$



Example 2. A system governed by an equation of the Van der Pol type is considered; the equations of state are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = (1-x_1^2)x_2 - x_1 + u \tag{45}$$

and the performance index is $J = \frac{1}{2}\int_0^\infty (x_1^2+x_2^2+u^2)dt$. The target set is the origin.

The control laws derived by use of the various methods are summarized in Table 2. The parameter optimization solution was calculated in [6] for initial conditions $x(0) = 1.75$ and $x_2(0) = -2.0$. This problem was also solved by equivalent linearization in [7]. The values of the performance index obtained by use of the various control laws are also presented in Table 2. The perturbation scheme gives the best results; however, both the combination and equivalent linearization procedures give results which are almost as good, and these methods involve considerably less calculation. The linearization method also gives good results. On the other hand, the parameter optimization technique appears to give the poorest results and is computationally the most difficult.

Typical solutions are compared in Fig. 2. In Fig. 2a, a phase-plane plot is presented. All of the methods except parameter optimization give trajectories which are so close to one another as to be indistinguishable. In

Fig. 2b, the various control laws are plotted versus time. It is seen that in constrast to example 1, the controls are close to one another, and the largest variations occur in the first second of operation.

Table 2

Control Law	Method	Performance Index Initial Conditions $x_1=.5$, $x_2=.5$	Performance Index Initial Conditions $x_1=1.0$, $x_2=1.0$
$-0.414x_1$ $-2.685x_2$	Linear	.8008	2.7026
$-0.414x_1-2.685x_2$ $+1.586x_1^2x_2$ $-0.194x_1^4x_2$	Combination	.7982	2.6335
$-0.414x_1+[1-x_1)^2$ $-\sqrt{2.828-x_1^2}]x_2$	Equivalent Linearization	.7977	2.6188
$-1.422x_1-3.808x_2$ $+0.2192x_2^3$	Parameter Optimization	.9194	3.1259
$-0.414x_1$ $-2.685+1.086x_1^2x_2$ $+0.583x_1x_2+0.072x_2^3$	Perturbation	.7971	2.5731

Fig. 2a

VAN DER POLS EQUATION — PHASE PLANE PLOT

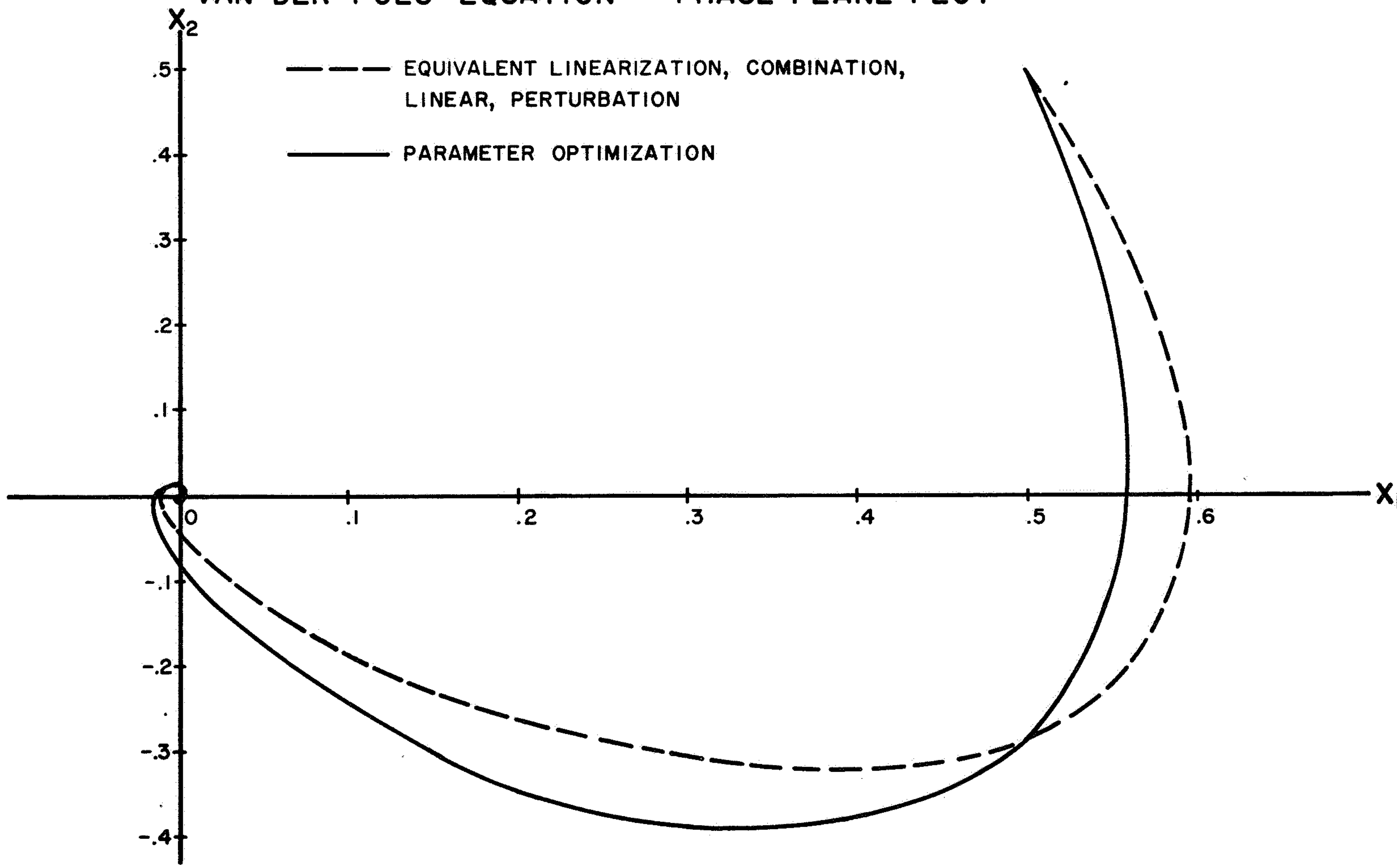
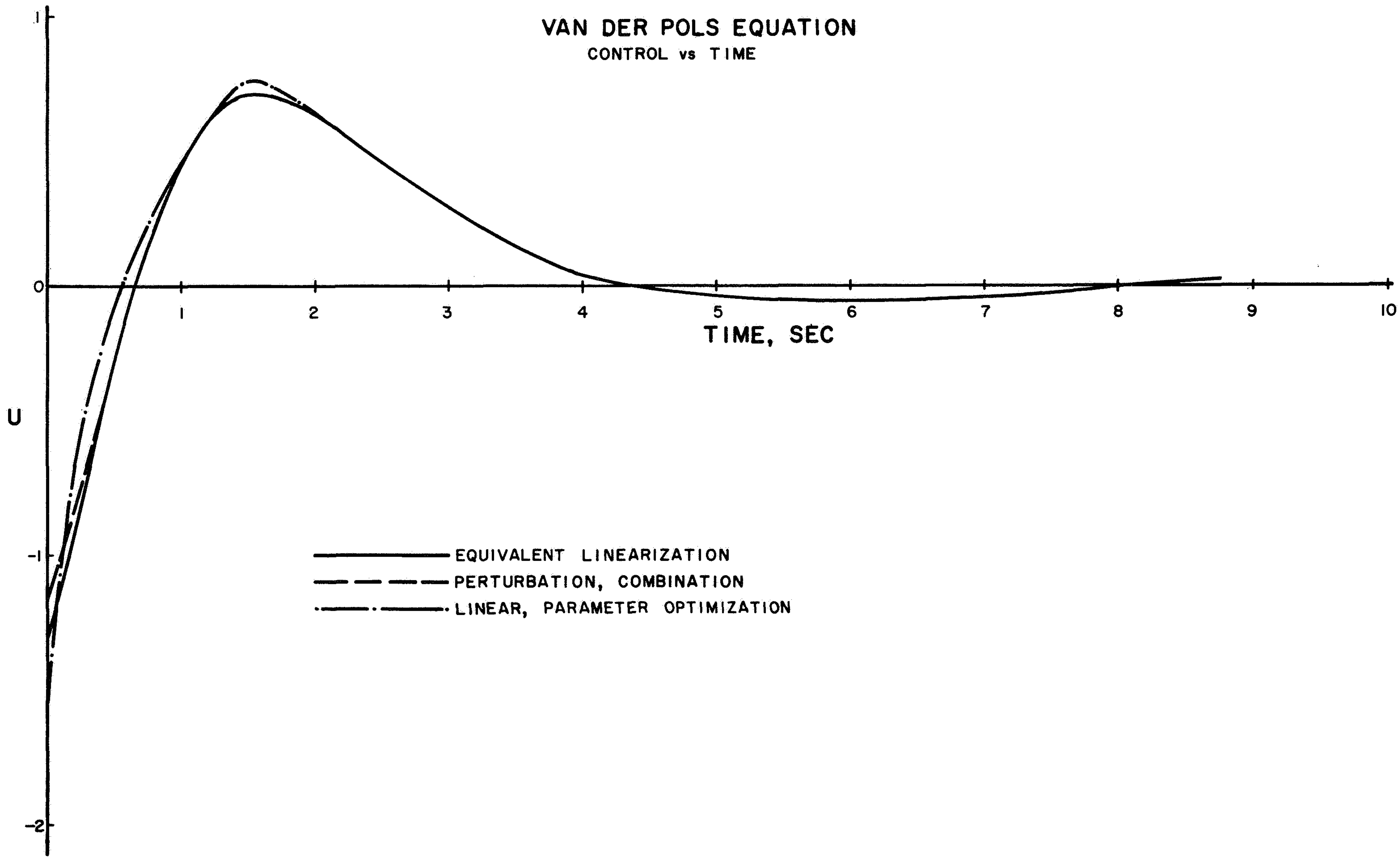


Fig. 2b



VAN DER POLS EQUATION
CONTROL vs TIME
.1
0
-1
-2
U
1
2
3
4
5
6
7
8
9
10
TIME, SEC
EQUIVALENT LINEARIZATION
PERTURBATION, COMBINATION
LINEAR, PARAMETER OPTIMIZATION

Example 3. The final example is concerned with the synthesis of the control logic for a regulator system using a field-controlled, D-C motor as a source of control torque. The equations governing such a system are

$$J\ddot{\theta} + C\dot{\theta} = K_T \frac{E}{R_a} i_f - \frac{K_T K_V}{R_a} \dot{\theta} i_f^2$$
$$L_f \frac{di_f}{dt} + R_f i_f = e_f \tag{46}$$

where J = moment of inertia of the load, C = coefficient of viscous damping, $\dot{\theta}$ = angular velocity of load, E = constant voltage applied across armature, R_a = armature resistence, R_f = field resistence, L_f = field inductance, i_f = field current, e_f = voltage applied across field terminals, K_v = motor voltage constant, and K_T = motor torque constant. Usually R_a is assumed large and the $\frac{K_T K_V}{R_a} \dot{\theta} i_f^2$ term is neglected; however, as is shown this term can affect performance and consequently is included in the analysis. Equation (46) can be written

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 + dx_3 + \varepsilon x_2 x_3^2 \\ \dot{x}_3 &= -Tx_3 + bu \end{aligned} \tag{47}$$

where $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = i_f$, $e_f = u$, $a = \frac{C}{J}$, $d = \frac{K_T E}{R_a J}$,

$\varepsilon = \frac{K_T K_V}{R_a J}$, $T = \frac{R_f}{L_f}$, and $b = \frac{1}{L_f}$. We wish to choos u so as to be a solution to the suboptimal control problem associated with this minimization of $J = \int_0^\infty (x'Qx+U'RU)dt$. Q is a positive definite, diagonal matrix, and S is the zero vector. Since the system is third order, it is virtually impossible to apply equivalent linearization and obtain analytical results. Also, the amount of work in applying perturbation or parameter optimization techniques is prohibitive. Consequently, the control will be synthesized first by finding the optimal control for the linear portion of the system and then applying the combination technique. For the following choice of parameters in (47),

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -.2 & 2 \\ 0 & 0 & 5 \end{bmatrix} , \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

$$R = Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

and $\varepsilon = 0.2$, the linearization procedure yields

$$u = -0.8722x_3 - 2.1206x_2 - 1.000x_2 .$$

Taking into account the system nonlinearity and applying the combination technique

$$u = -0.8722x_3 - 2.1206x_2 - 1.000x_1$$

$$+ 0.2((0.905x_1^2 - 66.825x_2^2 + 0.9049x_1x_2)x_3$$

$$+ 1.8098x_3^3 + 9.666x_2x_3^2) \ .$$

Results of a computer simulation of the above system are summarized in Table 3. It may be seen that the combination method gives the smallest value of the performance index; however, the linear control is of considerably simpler form. From Fig. 3a and Fig. 3b, graphs of angular error versus time and angular velocity error versus time respectively, it is seen that the linear procedure gives "closer" control, but Fig. 3c shows that the linear method demands a larger control voltage than does the combination method.

	PERFORMANCE INDEX		
	Initial Conditions		
Method of Control	$x_1=.1$ $x_3=.1$ $x_2=.1$	$x_1=.25$ $x_3=.1$ $x_2=.25$	$x_1=0.5$ $x_3=0.1$ $x_2=0.5$
Linear	.2293	1.1297	4.1941
Combination	.2288	1.1259	4.0627

TABLE 3

Fig. 3a

FIELD CONTROLLED D-C MOTOR
θ vs TIME

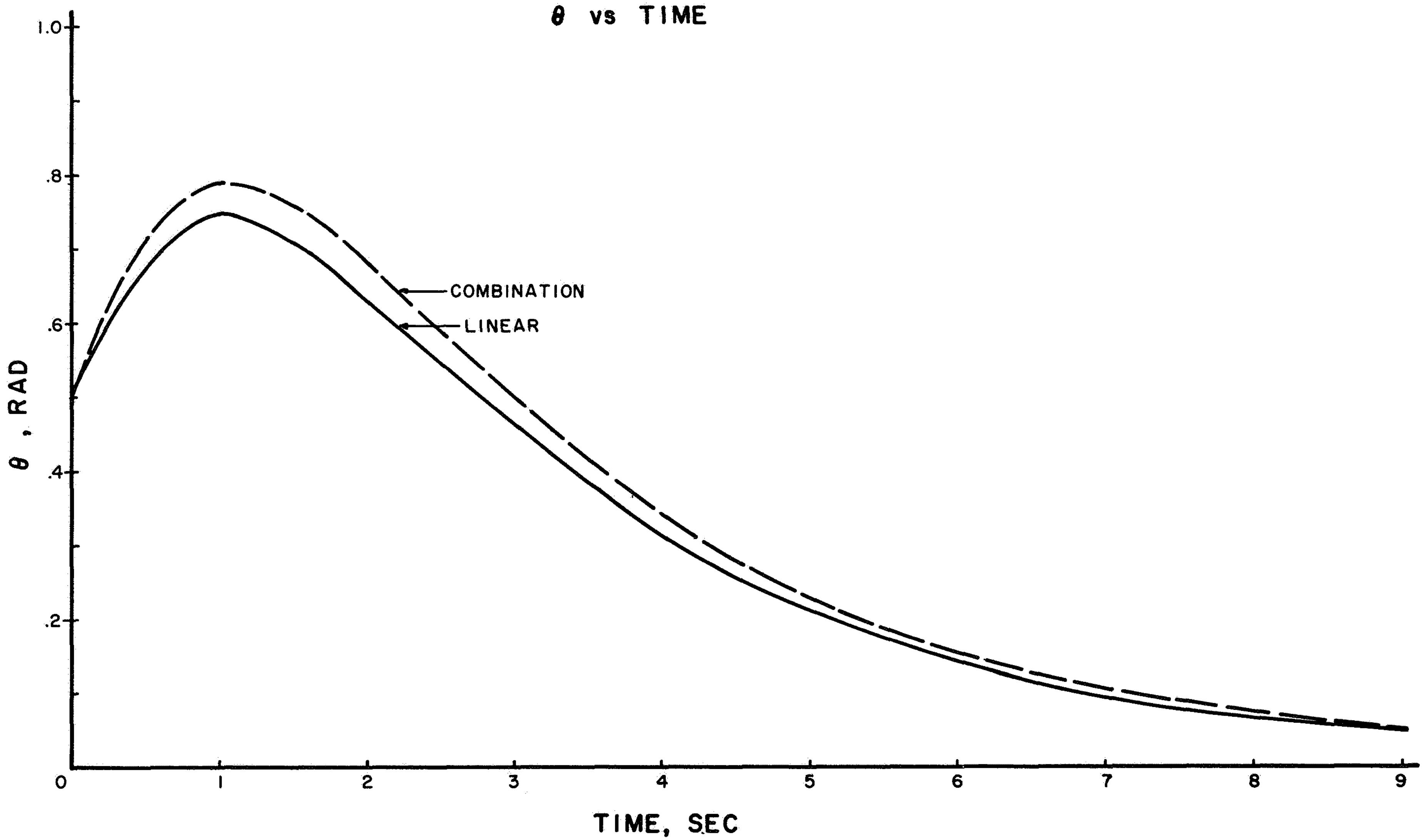
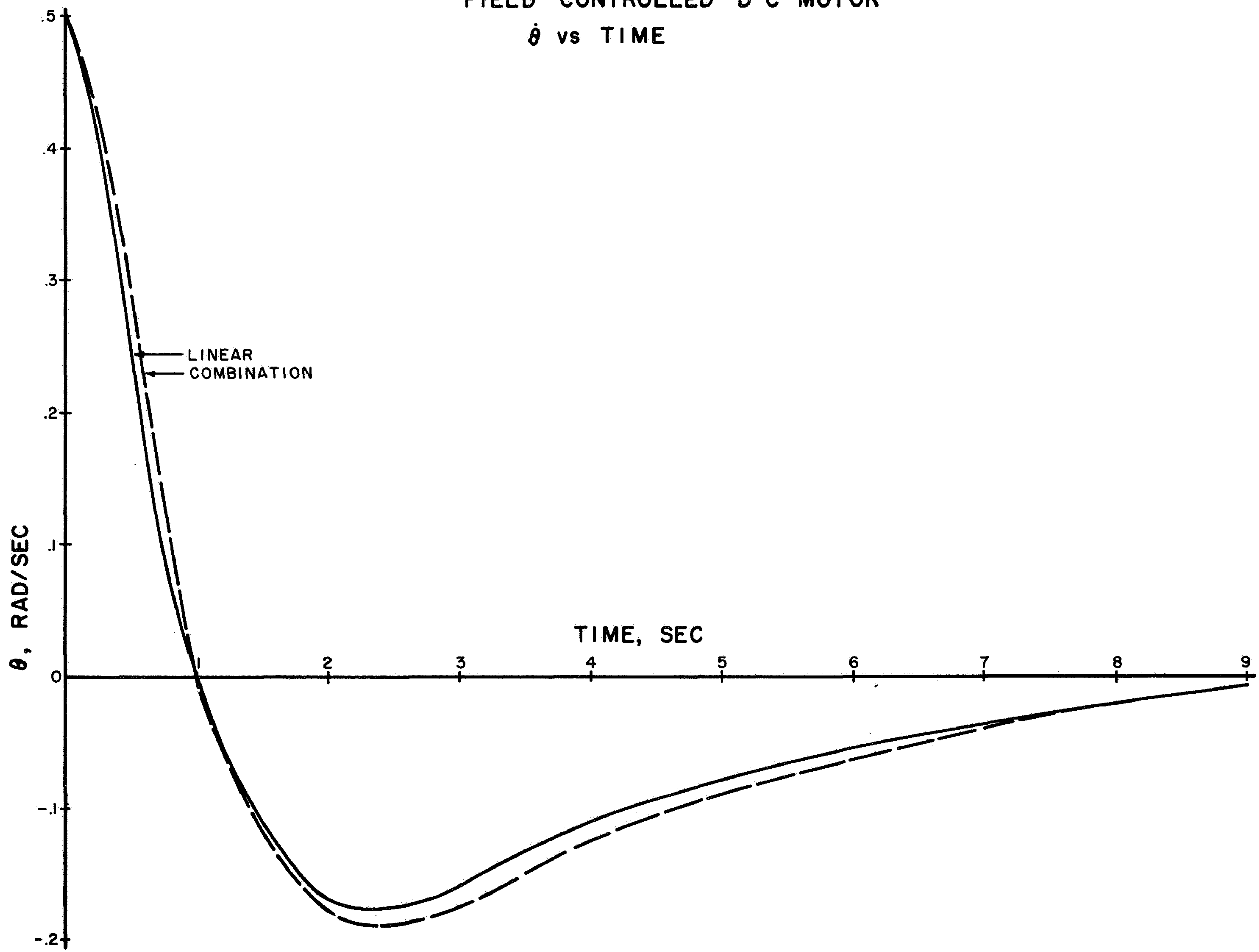


Fig. 3b

FIELD CONTROLLED D-C MOTOR

$\dot{\theta}$ vs TIME



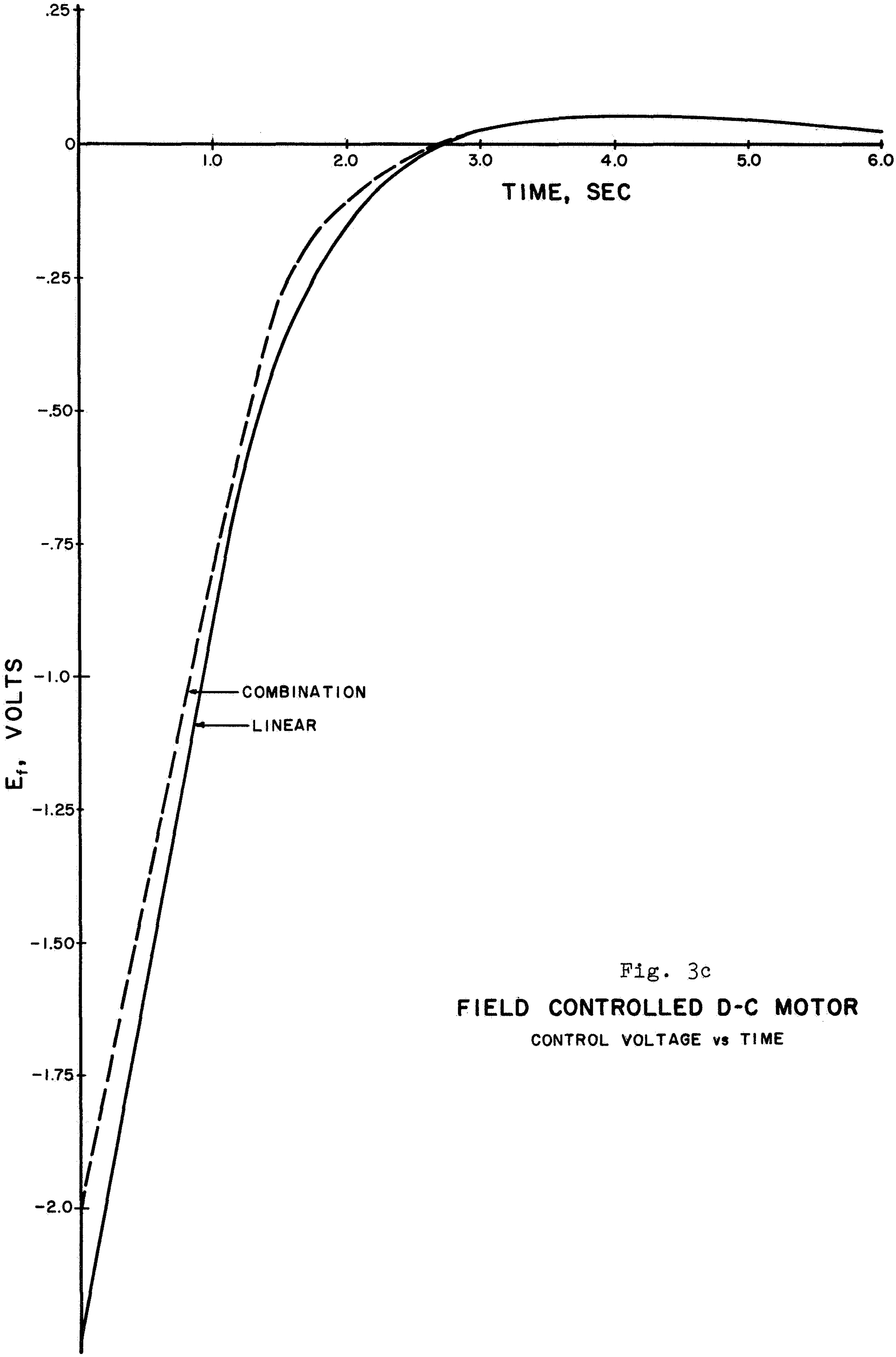


Fig. 3c

FIELD CONTROLLED D-C MOTOR

CONTROL VOLTAGE vs TIME

7. CONCLUSIONS

The linearization method gives surprisingly good results and is simple in form. On the other hand, the parameter optimization methods give poor values for the performance index for initial conditions greatly different from those for which the control law was originally computed. The equivalent linearization technique gives good results but is extremely difficult to use for systems of higher than second order, since for a system of order n the technique requires the analytic solution of a set of $\frac{n^2+n}{2}$ nonlinear, algebraic equations. The perturbation method also gives good results but is computationally unhandy. The results given by the combination method compare favorably with those obtained by the perturbation and equivalent linearization methods, and furthermore, use of the combination method is computationally feasible for systems of high order. On the other hand, the method sometimes gives complex control laws with little or no decrease in the performance index when compared to the linearization method. In the last analysis, choice of a control law depends upon the range of initial conditions expected, the simplicity of control law desired, and the importance of minimizing the performance index. However, the combination method seems to hold promise as a method for solving a large class of optimization problems.

Nothing has been said regarding the convergence properties of the various suboptimal control schemes. First, little is known about their convergence properties. Convergence to the optimal control in some small region about the origin has been established only for the perturbation method. Second, convergence is not of primary importance, since in practice, only a few terms can be used in choosing the suboptimal control. An additional point that has not been considered in detail is the satisfaction of the terminal constraint, that is, the transfer of the state to the target set. In general, the terminal constraint will not be satisfied for all initial conditions, so any suboptimal control would be limited to the set of initial states for which the terminal conditions is satisfied. In the case where the final time approaches infinity, the suboptimal control is valid only for initial conditions in the domain of asymptotic stability.

The problem considered in this paper has been the transfer of an initial state to some set containing the origin, using a feedback control. This problem has many applications in its own right; however, the techniques which have been considered in this paper are also applicable in determining a feedback controller which operates in a neighborhood of a reference, open-loop optimal trajectory [11].

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